Chapter 7.2: Exponential Change and Separable Differential Equations

Differential Equations

Sometimes it is easier to figure out how the function is changing than the original function.

Goal: Given information about y' and $y(a) = c$, determine $y(x)$.

Example: Give $y' = \frac{1}{x^2+1}$ and $y(1) = 2$, find $y(x)$.

First we compute the antiderivative.

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

Now we know $y(x) = \arctan x + C$ for some C. We try to compute C from $y(1) = 2$

$$2 = \arctan 1 + C$$
$$C = 2 - \frac{\pi}{4}$$

$$y(x) = \arctan x + 2 - \frac{\pi}{4}$$

More Interesting

Case $y' = f(x)$ is easy. $\quad y = C + \int f(x)\,dx.$

But what about

$$y' = (\text{stuff with } x \text{ and } y)$$

Hard, there are courses focusing on this.
Separable differential equation: isolate x and y

$$\boxed{\frac{dy}{dx} = y' = f(x) \cdot g(y)}$$

Solution plan

- Move y and y' to one side and x to the other side.

$$\frac{1}{g(y)} y' = f(x)$$

- Integrate both sides with respect to x
- Solve y and C using initial condition.

Rabbits like more rabbits. In particular, suppose there are initially 20 rabbits and the rate of change of population is $\frac{1}{10}$ of the current population. Determine the population as a function of time.

Let y be the number of rabbits at time t. We see $y(0) = 20$. The rate of change being $\frac{1}{10}$ means

$$\frac{dy}{dt} = \frac{1}{10}y$$
$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{10}$$
$$\frac{1}{y}\,dy = \frac{1}{10}\,dt$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{10}\,dt$$
$$\ln(y) = \frac{1}{10}t + C$$
$$y = e^{\frac{1}{10}t+C}$$
$$y = e^{C} \cdot e^{\frac{1}{10}t} = A \cdot e^{\frac{1}{10}t}$$

$$\frac{1}{y(t)}y(t)' = \frac{1}{10}$$
$$\int \frac{1}{y(t)}y(t)'\,dt = \int \frac{1}{10}\,dt + C$$
$$y(t) = u \text{ and u du}$$
$$\int \frac{1}{u}\,du = \int \frac{1}{10}\,dt + C$$
$$\ln(y) = \ln(u) = \frac{1}{10}t + C$$

$$20 = A \cdot e^{\frac{1}{10}\cdot 0}$$

$$y = 20e^{\frac{1}{10}t}$$

Separable equations

$$\boxed{\frac{dy}{dx} = f(x) \cdot g(y)}$$

We find solution using

$$\boxed{\int \frac{1}{g(y)}\, dy = \int f(x)\, dx}$$

Special form $\frac{dy}{dx} = k \cdot y$ and $y(0) = y_0$:

$\int \frac{1}{y}\, dy = \int k\, dx$

$\ln(y) = kx + C$

$y = e^{kx+C}$

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$y = e^{C} \cdot e^{kx}$

$y = A \cdot e^{kx}$

Now $y_0 = y(0) = A \cdot e^{k\cdot 0} = A$

Solution: $y = y_0 \cdot e^{k\cdot x}$

exponential growth or exponential decay

$$\frac{dy}{dx} = (1+y)e^x \text{ and } y(0) = 1$$

$$\begin{aligned}
\frac{dy}{dx} &= (1+y)e^x \\
\int \frac{1}{1+y}\, dy &= \int e^x\, dx \\
\ln(1+y) &= e^x + C \\
1 + y &= Ae^{e^x} \\
y &= Ae^{e^x} - 1.
\end{aligned}$$

Now solving for A:

$$\begin{aligned}
1 &= Ae^{e^0} - 1 \\
2 &= Ae \\
A &= \frac{2}{e}
\end{aligned}$$

Solution is $y = \frac{2}{e} \cdot e^{e^x} - 1$

Solve the following separable differential equations.

$\frac{dy}{dx} = e^{x-y}$ and $y(0) = 0$

$$\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$$
$$\int e^y \, dy = \int e^x \, dx$$
$$e^y = e^x + C$$
$$y = \ln(e^x + C)$$

Now solving for C:

$$0 = \ln(e^0 + C)$$
$$1 = 1 + C$$
$$C = 0$$

Solution is

$$y = \ln(e^x - 0) = x.$$

$\sqrt{2xy} \cdot \frac{dy}{dx} = 1$ and $y(2) = 0$

$$\sqrt{2xy}\frac{dy}{dx} = 1$$
$$\int \sqrt{2}\sqrt{y} \, dy = \int x^{-1/2} \, dx$$
$$\sqrt{2} \cdot \frac{2}{3} y^{3/2} = 2\sqrt{x} + C$$
$$y^{3/2} = \frac{3}{\sqrt{2}}\sqrt{x} + C$$
$$y = \left(\frac{3}{\sqrt{2}}\sqrt{x} + C\right)^{2/3}$$

Now solving for C:

$$0 = \left(\frac{3}{\sqrt{2}}\sqrt{2} + C\right)^{2/3}$$
$$C = -3$$

Solution is $y = \left(\frac{3}{\sqrt{2}}\sqrt{x} - 3\right)^{2/3}$.

The half-life of the plutonium-239 is 24360 years. Suppose that we begin with 10g of plutonium-239, find an equation that models that amount of plutonium as a function of years. (Hint: Use exponential growth/decay)

Let $y(t)$ be the amount of plutonium (in grams) as a function of time t (in years).

As this population grows proportional to its size, we know that

$$y(t) = y_0 e^{kt},$$

where k is some constant. As we begin with 10 grams, we know that $y_0 = 10$ and so the model becomes

$$y(t) = 10e^{kt}$$

Moreover, we know that the population halves after 24360, i.e.

$$\begin{aligned} 5 = y(24360) &= 10e^{24360k} \\ \ln(1/2) &= 24360k \\ \frac{\ln(1/2)}{24360} &= k \\ -\frac{\ln(2)}{24360} &= k \end{aligned}$$

Therefore

$$y(t) = 10e^{-\ln(2)\cdot(t/23460)}$$